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From these data the following points are to be noted:

1. The vital index has the lowest mean value in the quarter ending on March 31, the winter quarter. In that period the birth incidence is relatively low and the death incidence relatively high.
2. Next in value to this, but standing 9.40 ± 1.75 points above it, is the mean vital index for the autumn quarter ending December 31. The difference being 5.4 times its probable error, may be regarded as significant.
3. The spring quarter, ending June 30, shows the next higher mean, being 13.43 ± 1.75 points above the winter quarter.
4. The highest value of the index falls in the summer quarter, when births are most frequent and deaths least so. The mean value, however, lies only 5.54 ± 2.05 above that for the spring quarter, a difference which cannot be regarded as significant.
5. In variability of the vital index, the first two and the last quarters of the year, all exhibit significantly the same status. The vital index is distinctively more variable in the summer quarter, the difference in standard deviations when this quarter is compared with that ending June 30, amounting to 6.04 ± 1.45. This may be regarded as significant, being 4.2 times its probable error.

It thus appears that the extremely close compensatory relation between birth rate and death rate, which Pearl and Burger[3] have shown to hold in annual figures, does not obtain within the single year. Instead there is a well-marked statistically significant intra-annual, or seasonal fluctuation of the birth-death ratio.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 51.

[2] Cf. particularly Pearl, R., "The Vitality of the Peoples of America," *Amer. J. Hyg.*, **1,** 1921 (592–674).

[3] Pearl, R. and Burger, M. H., "The Vital Index of the Population of England and Wales, 1838–1920," *Proc. Nat. Acad. Sci.*, **8,** 1922, pp. 71–76.

A SOLUTION OF THE LINEAR MATRIX EQUATION BY DOUBLE MULTIPLICATION

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1. *Methods of Sylvester and of Maclagan Wedderburn.*—A matrix A of order N is defined as a binary assemblage of N^2 elements a_{ik}, where either subscript may have any value from 1 to N. We add two matrices by adding corresponding elements. We multiply by the rule

$$(AB)_{ik} = \Sigma a_{is} b_{sk};\ (s = 1, 2, \ldots, N). \qquad (1)$$

If $A_1, A_2, \ldots, A_h$ and $B_1, B_2, \ldots, B_h$ and C are all known matrices, while x is a required matrix, the linear matrix equation may be written

$$\theta(x) = A_1xB_1 + A_2xB_2 + \ldots + A_hxB_h = C. \qquad (2)$$

In 1884 Sylvester attacked this equation with characteristic energy.[1] He pointed out that, since each element of the matrix $\theta(x)$ is a linear function of the elements of x—the equation (2) being thus equivalent to N^2 linear equations of ordinary algebra—it follows θ has the properties of a matrix of order N^2 and satisfies a Hamilton-Cayley equation

$$\theta^n - m_1\theta^{n-1} + m_2\theta^{n-2} - \ldots + (-1)^n m_n I = 0 \qquad (3)$$

where $n = N^2$ and where I denotes the matrix of order N^2 whose elements I_{ik} are unity when subscripts are equal, otherwise zero. If we know the coefficients $m_1 \ldots m_n$ we can solve (2), as is evident on multiplying (2) and (3) through by θ^{-1}. We can find all the coefficients if we know the rule for finding m_n which is the determinant of the matrix θ; for the left side of (3) may be found by forming the determinant of $gI - \theta$ and afterwards putting θ in place of g. *To Sylvester, solution of the equation (2) was equivalent to finding, in terms of the matrices $A_1 \ldots A_h$ and $B_1 \ldots B_h$ the determinant of order N^2 belonging to the operation θ.*

His general method consists in setting up the two matrices of order N and of extent (étendue) h

$$x_1A_1 + x_2A_2 + \ldots + x_hA_h, \; x_1B_1 + x_2B_2 + \ldots + x_hB_h$$

where x_1, x_2, etc. are scalar variables. By forming the determinants of these two matrices we have two quantics in the variables. He shows that the required determinant is a *contrariant* called the *nivellant* of these two quantics, and is an integral but not necessarily a rational function of the coefficients of the quantics. These coefficients are regarded as known quantities. The formation of the contrariant is in general difficult. He works out the solution for several special cases. With regard to the general case he remarks[2]

"Pour présenter l'expression générale de ce déterminant pour une matrice d'un ordre et d'une étendue quelconques, c'est-à-dire pour résoudre l'équation linéaire en matrices dans toute sa généralité, il faudrait avoir une connaissance des propriétés des formes qui va beaucoup au delà des limites des facultés humaines, telles qu'elles se sont manifestées jusqu'au temps actuel, et qui, dans mon jugement, ne peut appartenir qu'à l'intelligence suprême."

But the indeterminate product of Gibbs was at that time unborn, or at least very young.

In 1904 Maclagan Wedderburn[3] gave a solution of (2) by infinite series. His method consists essentially in selecting one of the known matrices, as A_j, multiplying *by* A_j^{-1} and *into* B_j^{-1} giving (2) in the form

$$x + \psi(x) = A_j^{-1}.CB_j^{-1} = C'$$

where ψ is similar in form to θ but contains h-1 terms. Then

$$x = (1 + \psi)^{-1}\, C' = (1 - \psi + \psi^2 \ldots)C'.$$

The choice of the term A_jxB_j to be freed from matrix coefficients affects the convergence of the solution.

2. *Transformation from Matrices to Double Dyadics.*—By Gibbs' double multiplication we may set up explicit formulas for the coefficients $m_1 \ldots m_n$ of the Hamilton-Cayley equation (3). It is known that any matrix A is equivalent to a dyadic $\Sigma\mathbf{aa}'$ where $\mathbf{a}$ and $\mathbf{a}'$ are vectors. Any matrix term of the form AxB is equivalent to a sum of terms of the form $\mathbf{aa}'.\mathbf{X}.\mathbf{bb}'$ where $\mathbf{X}$ is the dyadic equivalent to x. We now introduce the double dot product defined by the rule $\mathbf{ab}:\mathbf{xy} = \mathbf{a}.\mathbf{xb}.\mathbf{y}$, and may write

$$\mathbf{aa}'.\mathbf{X}.\mathbf{bb}' = \mathbf{ab}'\mathbf{a}'\mathbf{b}:\mathbf{X}. \qquad (4)$$

Any term AxB is therefore equivalent to a sum of terms of the form $\mathbf{MN}:\mathbf{X}$ where $\mathbf{M}$ and $\mathbf{N}$ are dyadics, and the linear matrix equation (2) is equivalent to

$$\varphi:\mathbf{X} = \Sigma\mathbf{MN}:\mathbf{X} = \mathbf{C} \qquad (5)$$

where φ is a dyadic whose antecedents and consequents are dyadics, say a *double dyadic*.

3. *Dyadics Treated Like Vectors in Space of Higher Dimensions.*—Adopting unit vectors $\mathbf{e}_1 \ldots \mathbf{e}_n$ such that $\mathbf{e}_i.\mathbf{e}_k$ is unity when subscripts are equal, otherwise zero, any dyadic $\mathbf{M}$ may be expanded in terms of the dyads $\mathbf{e}_i\mathbf{e}_k$; and if we agree on some definite order, no matter what, among these dyads we may set $\mathbf{E}_r = \mathbf{e}_i\mathbf{e}_k$, *lettering r run from 1 to N^2 as i and k each run from 1 to N.* With $n = N^2$, any dyadic $\mathbf{M}_t$ may be expanded

$$\mathbf{M}_t = \mathbf{E}_1 m_{t1} + \mathbf{E}_2 m_{t2} + \ldots + \mathbf{E}_n m_{tn} \qquad (6)$$

where m_{t1} etc. are scalars. The double dyadic φ may be expanded as

$$\varphi = \mathbf{E}_1\mathbf{M}_1 + \mathbf{E}_2\mathbf{M}_2 + \ldots + \mathbf{E}_n\mathbf{M}_n. \qquad (7)$$

Now $\mathbf{e}_i\mathbf{e}_k:\mathbf{e}_r\mathbf{e}_s$ is unity when *both $i = r$ and $k = s$*, otherwise zero. This is the same assaying $\mathbf{E}_p:\mathbf{E}_q$ is unity when subscripts are equal, otherwise zero. It is immediately evident that the double dyadic φ will behave with reference to double dot product as would an ordinary dyadic in n- dimensional space with reference to ordinary dot product, and equally evident that the whole of the usual matrix theory will apply to φ with no essential change.

4. *Formation of the Hamilton-Cayley Equation.*—Since $\varphi:\mathbf{X} = \theta(x)$ the Hamilton-Cayley equation for φ will be the same as that for θ. It remains to show how to calculate the coefficients $m_1 \ldots m_n$ of this equation. Let $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_p$ and $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_p$ be any dyadics:

Definition.—The scalar $((\mathbf{M}_1\mathbf{N}_1, \mathbf{M}_2\mathbf{N}_2, \ldots, \mathbf{M}_p\mathbf{N}_p))_S$ is defined to be the sum of terms computed as follows: the leading term is the product of double dot products $\mathbf{M}_1:\mathbf{N}_1\mathbf{M}_2:\mathbf{N}_2 \ldots\ldots\ldots \mathbf{M}_p:\mathbf{N}_p$; the other terms are of like form and are obtained from the leading term by keeping the antecedents $\mathbf{M}_1 \ldots\ldots \mathbf{M}_p$ fixed in position while the consequents $\mathbf{N}_1 \ldots \mathbf{N}_p$ are permuted in all ways. The sign of any term is plus or minus according as the number of simple interchanges needed to form that term is even or odd.

Thus if $p = 2$ we shall have

$$((\mathbf{M}_1\mathbf{N}_1,\ \mathbf{M}_2\mathbf{N}_2))_S = \mathbf{M}_1:\mathbf{N}_1\mathbf{M}_2:\mathbf{N}_2 - \mathbf{M}_1:\mathbf{N}_2\mathbf{M}_2:\mathbf{N}_1.$$

If $\varphi_1, \varphi_2, \ldots, \varphi_p$ are double dyadics defined as in (7) the scalar $((\varphi_1, \varphi_2, \ldots, \varphi_p))_S$ is defined to be the result of expanding each φ as in (7), multiplying out term by term, and adding the scalars of all the terms of the product.

If each $\mathbf{M}$ be expanded as in (6) the scalar elements of every φ will form a square matrix m_{ts} of order N^2 or n, *so that each* φ *depends on* N^4 *scalar elements*. The scalar $((\varphi_1, \varphi_2, \ldots, \varphi_p))_S$ is the sum of cubic determinants of order p whose main diagonals lie on the main diagonals of the respective matrices m_{ts}; the p matrices m_{ts} are to be regarded as p nonsignant layers of a cubic matrix.[4]

We may now suppose these p double dyadics to be all equal. The scalar just written becomes $((\varphi, \varphi, \ldots \text{to } p \text{ factors}))_S$ and may be abbreviated $((\varphi^p))_S$.

Theorem.—The scalar $((\varphi^p))_S = p!m_p$ where m_p is the coefficient of φ^{n-p} in the Hamilton-Cayley equation for φ.

Proof. Let φ be expanded as in (7). The terms of the indeterminate product $(\varphi, \varphi, \ldots$ to p factors) are of the form

$$(\mathbf{E}_i\mathbf{M}_i,\ \mathbf{E}_j\mathbf{M}_j,\ \mathbf{E}_k\mathbf{M}_k,\ \ldots\ \mathbf{E}_r\mathbf{M}_r) \qquad (8)$$

where any subscript will have any value from 1 to n. Since $\mathbf{E}_s:\mathbf{M}_t = m_{ts}$ the scalar of the expression (8) has for its leading term $m_{ii}m_{jj}m_{kk} \ldots m_{rr}$ and, by the definition, the scalar is the determinant of order p of which this is the leading term, a minor of the determinant of φ. Thus the scalars of all expressions (8) vanish except when the subscripts $i, j, k, \ldots, r$ are all different. Each choice of subscripts $i, j, k, \ldots, r$ will occur $p!$ times. The scalar $((\varphi^p))_S$ is therefore $p!$ times the sum of minors of order p taken along the main diagonal of the determinant of φ. By the usual matrix theory then $((\varphi^p))_S = p!m_p$ as was to be proved.[5]

5. *Transformation Back to Matrices.*—It has thus been shown how the invariants $m_1 \ldots m_n$ of the Hamilton-Cayley equation for φ or θ may be calculated by products which are closely analogous with Gibbs' products of ordinary dyadics. To complete the solution we need to see how these invariants may be formed directly from the given matrices $A_1 \ldots A_h$ and $B_1 \ldots B_h$ without the need of first forming φ by the rule (4).

Consider first m_1, the coefficient of φ^{n-1}. If $\varphi = \Sigma\mathbf{MN}$ we have $m_1 = \Sigma\mathbf{M}:\mathbf{N} = \Sigma\mathbf{ab}':\mathbf{a}'\mathbf{b} = \Sigma\mathbf{a}.\mathbf{a}'\mathbf{b}.\mathbf{b}' = \Sigma A_S B_S$ where A_S and B_S are the scalars of the dyadics corresponding to A and B, or, in matrix language, they are the respective sums of the elements of the main diagonals. Thus

$$m_1 = A_{1S}B_{1S} + A_{2S}B_{2S} + \ldots + A_{hS}B_{hS}. \qquad (9)$$

For m_2 we have, by the distributive character of all the steps involved in the process,

$$\begin{aligned} 2m_2 &= ((\varphi, \varphi))_S = \Sigma((\mathbf{M}_i\mathbf{N}_i, \mathbf{M}_k\mathbf{N}_k))_S \\ &= \Sigma((\mathbf{a}_i\mathbf{b}'_i\mathbf{a}'_i\mathbf{b}_i, \mathbf{a}_k\mathbf{b}'_k\mathbf{a}'_k\mathbf{b}_k))_S \\ &= \Sigma[(\mathbf{a}_i\mathbf{b}'_i\mathbf{:}\mathbf{a}'_i\mathbf{b}_i)\,(\mathbf{a}_k\mathbf{b}'_k\mathbf{:}\,\mathbf{a}'_k\mathbf{b}_k) - (\mathbf{a}_i\mathbf{b}'_i\mathbf{:}\,\mathbf{a}'_k\,\mathbf{b}_k)\,(\mathbf{a}_k\mathbf{b}'_k\mathbf{:}\,\mathbf{a}'_i\mathbf{b}_i)] \\ &= \Sigma[\mathbf{a}_i.\mathbf{a}'_i\mathbf{b}_i\mathbf{:}\,\mathbf{b}'_i\mathbf{a}_k.\mathbf{a}'_k\mathbf{b}_k.\mathbf{b}'_k - \mathbf{a}_i.\mathbf{a}'_k\mathbf{a}_k.\mathbf{a}'_i\mathbf{b}_k.\mathbf{b}'_i\mathbf{b}_i.\mathbf{b}'_k] \\ &= \Sigma[A_{iS}B_{iS}A_{kS}B_{kS} - (A_iA_k)_S(B_kB_i)_S] \end{aligned} \tag{10}$$

where, as before, A_{iS} is the sum of elements of the main diagonal of A_i, etc. and $(A_iA_k)_S$ is the sum of elements of the main diagonal of the product of the two matrices A_i and A_k.

By similar transformation we may show that

$$\begin{aligned} 6m_3 &= \Sigma[A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS} - A_{iS}B_{iS}(A_jA_k)_S(B_kB_j)_S \\ &\quad - A_{jS}B_{jS}(A_kA_i)_S(B_iB_k)_S - A_{kS}B_{kS}(A_iA_j)_S(B_jB_i)_S \\ &\quad + (A_iA_jA_k)_S(B_kB_jB_i)_S + (A_kA_jA_i)_S(B_iB_jB_k)_S]. \end{aligned} \tag{11}$$

In every case all subscripts run from 1 to h, *and terms where some or all of the subscripts are equal do not in general vanish.* For a matrix term AxB is not in general equivalent to a *single* dyadic term $\mathbf{MN:X}$.

The general rule for any m is now easy to see. The quantity $p!m_p$ may be written as a *summation of square bracket expressions, each square bracket enclosing $p!$ terms. The leading term is*

$$\Sigma A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS} \dots A_{rS}B_{rS}$$

where the p subscripts $i, j, k, \dots r$ are those of the given matrix coefficients in (2) and may be either alike or different. The other terms are formed by first keeping the antecedents A fixed and making interchanges among the consequents B. A multiple interchange such that q consequents $B_aB_bB_c \dots B_eB_fB_g$ (whether alike or not) change places among themselves leads to a product of two factors as $(A_cA_bA_e \dots A_dA_fA_a)_S$ $(B_aB_fB_d \dots B_eB_bB_c)_S$ where the order of consequents is the order they stand after the interchange, while the order of antecedents is the reverse of that of the consequents. Any antecedents or consequents which have not been moved yield the same scalars as in the leading term.

These quantities m may be transformed in many ways.

6. *A Second Method of Solution.*—Another method of solving the equation (2) follows at once from the analogy between θ and an ordinary dyadic. If $\varphi = \Sigma\mathbf{NM}$ let $\varphi' = \Sigma\mathbf{NM}$ corresponding to θ', that is to $\Sigma A'xB'$. *Let $C_1, C_2, \dots C_{n-1}$ be n-1 matrices or dyadics such that each is orthogonal to the given matrix C in the sense that $C:C_i = 0$ for $i = 1, 2, \dots, n-1$. Form $n-1$ new matrices $\theta'(C_i)$. A matrix orthogonal to all these new matrices will in general be a solution of (2) aside from a scalar factor which may be found by direct substitution.* This scalar may also be found as a function of m_n and the matrices $C, C_1, C_2, \dots C_{n-1}$.

This method is analogous with Hamilton's original method of solving the linear vector equation.[6]

7. *Conclusion.*—The analogy between a dyadic and a vector by virtue of the double dot product may be extended to polyadics and K-tuple dot

products, hence to dyadics whose antecedents and consequents are polyadics, say double polyadics. The analogy may also be applied to certain other products. Thus the scalar $((\mathbf{ab}, \mathbf{cd}))_S$ or $\mathbf{a.bc.d} - \mathbf{a.dc.b}$ may be regarded as a product of two dyadics $\mathbf{M}*\mathbf{N}$ and dyadics found which are orthogonal with respect to star product, that is such that $\mathbf{M}_i*\mathbf{N}_k$ is unity when subscripts are equal, otherwise zero. We may similarly have star products for double dyadics and double polyadics. We shall have Hamilton-Cayley equations with respect to star products, where the scalars m are formed by star product just as those of this paper were formed by double dot product. The present paper is intended as introduction to more detailed treatment of some of these phases of multiple algebra. They have a bearing on ordinary algebra by virtue of the fact that a polynomial of degree K in N variables corresponds to a symmetrical polyadic of order K in space of N dimensions.[7]

[1] In a series of *Paris* papers, *C. R. Acad. Sci.*, **98**, and **99** (1884) and elsewhere. The complete bibliography may be found in *Bull. Inter. Assoc. Promoting Study Quaternions and Allied Branches*, particularly for March, 1908.

[2] *Paris, C. R. Acad of Sci.*, **99**, p. 435.

[3] "Note on the Linear Matrix Equation," *Edinburgh Math. Soc. Proc.*, **22**, 1904 (49–53).

[4] The laws of signant and non-signant indices for p-way determinants were given by Lepine Hall Rice, *Amer. J. Math.*, **40**, No. 3, July, 1918.

[5] By forming the determinant of $\varphi - \lambda I$.

[6] *Elements of Quaternions*, Vol. I, 2nd. Ed., p. 498.

[7] Binary quadrics as vectors in 3-space, with what I here call star orthogonality, have been studied in great detail by Emil Waelsch. See the bibliographies referred to in Note 1. His method is given in *Wien. Ber.*, **112**, 1903, p. 645, 1091, and 1533, and called "Binäre-analyse."